Some mass transfer effects on the wall jet

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The wall jet with suction or injection is investigated; an analysis under conditions corresponding to similar flows is shown to reduce to an eigenvalue problem. Asymptotic solutions valid far from the surface are used to initiate the integration and circumvent the usual iteration associated with the two-point boundary-value problem. Typical solutions for various rates of suction and injection are obtained. It is found that the skin friction decreases with increasing rate of suction. Representative thermal solutions are obtained for Prandtl and Lewis numbers equal to one, under the special condition that the surface temperature is equal to the ambient temperature or that the enthalpy varies monotonically from the surface value to the ambient value.

1. Introduction

The term 'wall jet' has been introduced by Glauert (1956) to characterize the flow engendered by a jet blown tangentially or normal to a plane surface and spreading out over it, as depicted in figure 1. With the assumption of constant density, Glauert studied both radial and two-dimensional wall jets, and both laminar and turbulent flows, over an impermeable surface in an otherwise stationary atmosphere (see, for example, figures 1*a* and *b*). Attention was confined to regions far downstream of the origin of the jets. In these regions similar solutions were obtained explicitly for laminar flows. For turbulent flows, assumptions as to the nature of the eddy viscosity led to some reasonable predictions concerning the approximate similarity of the velocity distribution and the rate of growth or mixing of the viscous layer. Experimental results (Bakke 1957; George 1959; and Schwarz & Cosart 1961) have shown reasonable agreement with the predictions of Glauert concerning turbulent-profile similarity, growth and shape.

Extension of the aforementioned incompressible wall-jet solutions to the compressible régime has been carried out by Glauert (1957), Bloom & Steiger (1958), and Riley (1958). These investigations have dealt mainly with laminar, similar flows.

A class of boundary-layer flows which are not similar but can be considered to be associated with flows of wall-jet character has been studied by Bloom & Steiger (1961). These consist of wall jets under the influence of an outer flow with constant or variable velocity, as depicted in figures 1*c* and *d*. Bloom & Steiger treat the laminar, compressible case and, in essence, observe the effects of perturbing the known compressible wall-jet solution. The perturbation solution is achieved by a series-expansion technique, and the region of validity is restricted to flows wherein the external velocity is reasonably small compared to the maximum velocity in the viscous layer.

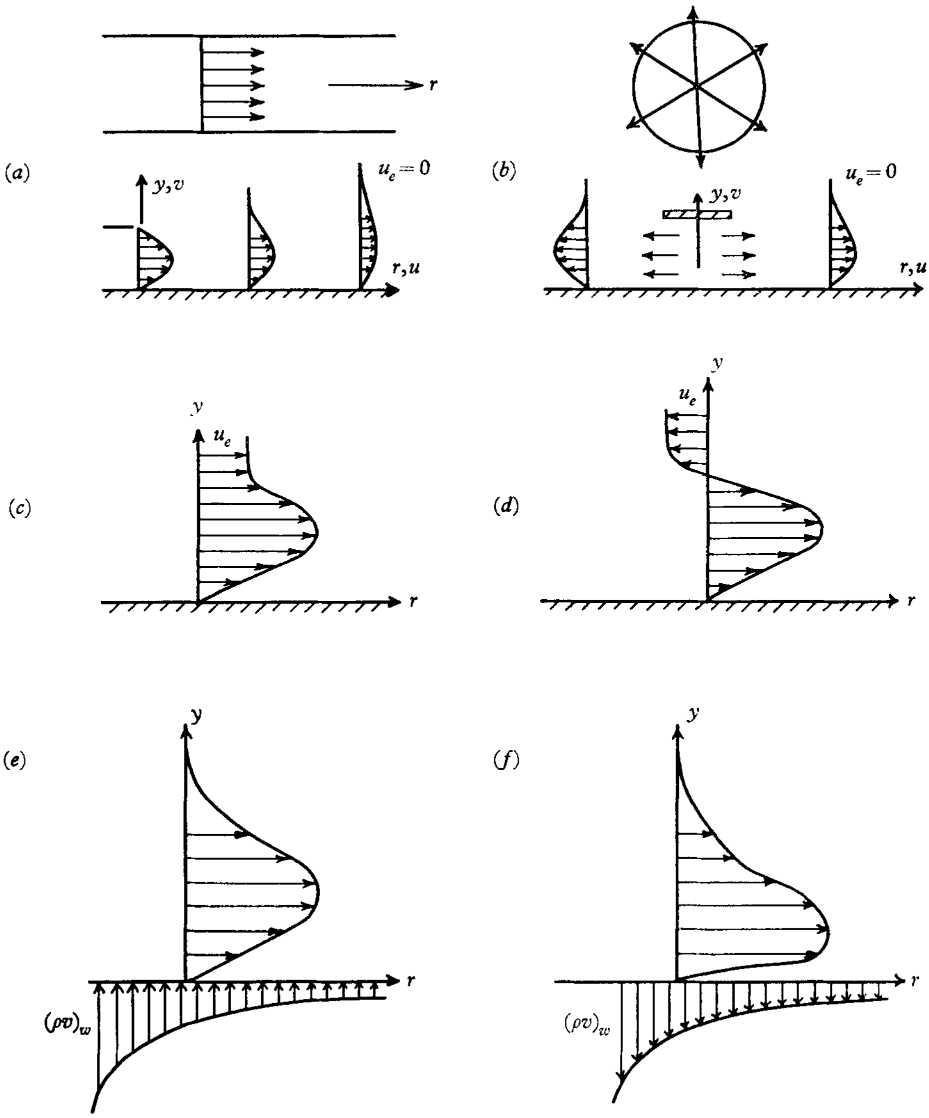


FIGURE 1. Schematic diagrams: (a) two-dimensional wall jet in a motionless ambient over an impermeable surface; (b) axisymmetric wall jet in a motionless ambient over an impermeable surface; (c) wall jet flow in the same direction as outer velocity u_e over an impermeable surface; (d) wall jet flow in the opposite direction as outer velocity u_e, and producing reverse flow, over an impermeable surface; (e) wall jet flow in motionless ambient over a permeable surface (injection); (f) wall jet flow in motionless ambient over a permeable surface (suction).

It should be noted that the wall-jet solutions are valid for the jet flow over cones, cylinders and wedges. In the case of a cone the radial wall-jet solution applies, provided that the jet thickness is small compared to the local transverse radii of curvature; the streamwise co-ordinate is simply interpreted as the distance along the cone generator. Likewise, it is evident that the two-dimensional wall-jet solutions can be applied to the flow along a wedge and the flow along the generators of cylinders whose transverse radii of curvature are large compared to the jet thickness. The effects of transverse curvature are being investigated at the present time; the influence of streamwise surface curvature on wall jet flows has not yet been studied.

This paper studies, by means of 'similar' solutions, the fluid dynamics and heat-transfer characteristics of a wall jet spreading out over a permeable plane surface in otherwise stationary ambient surroundings, as shown schematically in figures 1 *e* and *f*. Particular attention is given to the solution of the velocity field, where an eigenvalue problem is posed. Although the eigenvalues are restricted on mathematical grounds, it is shown that there still exist an infinite number of similar solutions with each eigenfunction requiring a specified distribution of normal velocity along the wall. The Crocco transformation (cf. Crocco 1946) and a new dependent variable proportional to the shear are employed, so that the equations are in a form convenient for numerical integration. The mass density-viscosity ratio is assumed constant. A procedure for handling the resulting two-point boundary value problem suggested by Libby (1962) and Fox & Libby (1962) is applied to the wall jet in which use is made of the asymptotic solutions.

In order to present some representative thermal solutions, the mixture is assumed to be homogeneous and non-reacting, the Prandtl and Lewis numbers are taken to be unity, and the surface enthalpy is taken to be equal to the ambient enthalpy, or the enthalpy is assumed to vary monotonically from the surface value to the ambient value.

2. The momentum equation

Consider the momentum equation arising after application of the Levy-Lees transformation (Lees 1956) to the boundary-layer equations with conditions of similarity imposed and with zero pressure gradient,

$$f_\alpha''' + f_\alpha f_\alpha'' - 2\alpha f_\alpha'^2 = 0, \tag{1}$$

where $\alpha = (\tilde{s}/u_r)(du_r/d\tilde{s}) = \text{const.}$ is an eigenvalue, f_α the associated eigenfunction, $u = u_r f_\alpha'$, $\rho\mu = \rho_r\mu_r$, and the transformation variables η and $\tilde{s}$ are defined by

$$\eta = [\rho_r u_r r^k/(2\tilde{s})^{\frac{1}{2}}]\int_0^y (\rho/\rho_r)\,dy, \quad \tilde{s} = \int_0^r \rho_r\mu_r u_r r^{2k}\,dr, \tag{2a, b}$$

with $k = 0$ or 1 for two-dimensional or radial flow, respectively. Here r is the space co-ordinate in the streamwise direction, y the normal co-ordinate, u and w are the velocities in the r and y directions, the subscript r represents a reference value and μ is the coefficient of viscosity.

In the case of the wall jet the boundary conditions associated with (1) are

$$f_\alpha(0) = f_{\alpha w}, \quad f_\alpha'(0) = f_\alpha'(\infty) = 0, \tag{3}$$

where $f_{\alpha w} \gtrless 0$ for suction or injection, and the subscript w refers to the condition at the wall.

In addition to the two-point boundary-value problem posed by (3), (1) also implies the existence of an eigenvalue problem. That is, for each value of α (which so far is unrestricted in the sense that it may take on any real value) only one solution f_α with the correct asymptotic behaviour as $\eta \to \infty$ can be found. Indeed, as $\eta \to \infty$, (1) has a solution of the form $f_\alpha = A \ln \eta + B + O(1/\eta) + \ldots$, where $A = A\{f''_\alpha(0), \alpha\}$ and B are constants. It is desirable to require that f'_α decay exponentially as $\eta \to \infty$, then $f_\alpha(\infty)$ must equal a constant which is possible only if $A = 0$. Two equations for the unique determination of $f''_\alpha(0)$ and α can be then obtained from (1), namely

$$f''_\alpha(0) = -(2\alpha+1)\int_0^\infty f'^2_\alpha\, d\eta \tag{4a}$$

and

$$f_\alpha(0)\, f''_\alpha(0) = -2(\alpha+1)\int_0^\infty f_\alpha f'^2_\alpha\, d\eta, \tag{4b}$$

where $f''_\alpha(\infty) = 0$ has been assumed. $(4a)$ is obtained by integrating (1) from $\eta = 0$ to $\eta = \infty$, while $(4b)$ is obtained by multiplying (1 by f_α and then integrating.

There are several interesting features associated with (4). If it is assumed that $f''_\alpha(0) \geqslant 0$, then $(4a)$ imposes the restriction $\alpha \leqslant -\frac{1}{2}$ since f'^2_α, and hence

$$\int_0^\eta f'^2_\alpha\, d\eta,$$

are always positive. Furthermore, $(4a)$ states that if $f''_\alpha(0) = 0$ (blow-off), then $\alpha = -\frac{1}{2}$; from $(4b)$ it follows that

$$\int_0^\infty f_{-\frac{1}{2}} f'^2_{-\frac{1}{2}}\, d\eta = 0,$$

hence $f_{-\frac{1}{2}}$ must be an asymmetric function varying from $f_{-\frac{1}{2}} = -f_{-\frac{1}{2}}(\infty)$ at the wall to $f_{-\frac{1}{2}} = f_{-\frac{1}{2}}(\infty)$ as $\eta \to \infty$. In addition, for $f_\alpha(0) = 0$ (impermeable surface), $(4b)$ yields $\alpha = -1$ and $(4a)$ yields

$$f''_{-1}(0) = \int_0^\infty f'^2_{-1}\, d\eta.$$

Clearly, a unique value of α is attained for each value of $f''_\alpha(0)$ and therefore for each solution. For brevity, the subscript α will henceforth be omitted. It is interesting to note the following group property exhibited by (1) and (3), first pointed out by Glauert (1956). If $f_0(\eta)$ is a solution then $f_1 = Af_0(A\eta)$ is also a solution for any value of the constant A.

In general, a solution to the systems (1) and (3) can only be achieved by numerical integration. However, there are three exceptions, namely, the cases where $\alpha = -1{\cdot}0$, $\alpha = -0{\cdot}5$, and $f = f_w = \text{const.}$ (α arbitrary). As previously noted, $\alpha = -1{\cdot}0$ corresponds to the flow along an impermeable surface (i.e. $f_w = 0$). Under this condition and with $f'(0) = f'(\infty) = 0$, Glauert obtained the following solution to (1):

$$f = g^2, \quad \eta = \ln\{(1+g+g^2)/(1-g)\}^{\frac{1}{2}} + \sqrt{3}\tan^{-1}\{\sqrt{3}g/(2+g)\}, \tag{5}$$

where, because of the group property, $f(\infty) = 1$ is assumed without loss in generality. For $\alpha = -0{\cdot}5$, (1) can be integrated to yield

$$f'' + ff' = \text{const.} \tag{6a}$$

However, since $f''(\infty) = f'(\infty) = 0$, the constant must also be zero. For (6*a*) to be valid at the wall, it follows that $f''_w = 0$; thus $\alpha = -\frac{1}{2}$ corresponds to blow-off. An additional integration of (6*a*) is possible and yields

$$f' + \tfrac{1}{2}f^2 = \tfrac{1}{2}f_w^2. \tag{6b}$$

It should be noted that (6) is similar in form to those equations arising from a study of the two-dimensional jet. However, for a complete matching it is necessary to change independent variables from η to, say, ζ, so that $\eta = 0$ corresponds to $\zeta = -\infty$ and $\eta = 0$ corresponds to $\zeta = +\infty$. A transformation of this type, which must necessarily retain the form of (6*b*), could not be found. In the third case, it is seen that $f = f_w = \text{const.}$ is a solution to (1) and satisfies (3). However, $f = \text{const.}$ requires that the streamwise velocity be identically zero everywhere, and therefore corresponds to a trivial solution.

Reduction to first-order equation and treatment of the two-point boundary-value problem

Following the method of Crocco (1946), the velocity ratio $f' = \xi$ is introduced as the independent variable and a shear function, $f'' = G$, is defined. Use of these variables in (1) results in

$$dG/d\xi = -f + 2\alpha\xi^2/G, \tag{7}$$

$$df/d\xi = \xi/G, \tag{8}$$

subject to the boundary conditions

$$\text{at } \xi = 0 \text{ (wall)}, \qquad f = f_w; \tag{9}$$

$$\text{as } \xi \to 0 \text{ (outer edge)}, \quad G \to 0. \tag{10}$$

Note that $\xi = 0$ at both end-points; however, as will be seen, this introduces no difficulty in the integration scheme employed.

The transformation to ξ as the independent variable and the introduction of G reduces (1) to a set of two non-linear, first-order differential equations which are amenable to standard techniques of numerical integration. However, an essential difficulty in numerical analysis is associated with the two-point nature of the boundary-value problem. The usual procedure for a given α is to initiate the integration at the wall choosing a value of f_w. An arbitrary value of G_w is chosen and the integration carried to the outer edge where in general the boundary condition is not satisfied. Guided iteration must then be performed on the value of G_w such that (10) is indeed satisfied. However, there is no assurance that the particular value of α selected will admit a solution. Thus, even for these simple equations the solution becomes quite formidable.

Libby (1962), and more recently Fox & Libby (1962), employed a different technique in treating a two-point boundary-value problem; they used the asymptotic solution valid as $\eta \to \infty$. Consider then the solution of (1) for η large using the following appropriate approximations:

$$f \simeq f_\infty + f_1, \quad f' \simeq f_1, \quad f'' \simeq f_1'', \tag{11}$$

where $f_\infty = \text{const.}$, $f_1 \ll f_\infty$, and $f_1', f_1'' \ll 1$. Insertion of (11) in (1), and neglecting products of f_1 and its derivatives results in

$$f_1''' + f_\infty f_1'' = 0. \tag{12}$$

Integration yields

$$f \simeq f_\infty - \xi/f_\infty, \quad G \simeq f_\infty \xi. \tag{13}$$

The asymptotic solution given by (13) may be employed as follows. Choose values of f_∞ and ξ with $\xi \ll f_\infty$ and compute values of f and G. The integration of (7) and (8) is then carried out for increasing values of ξ until $G \to 0$. At $G = 0^-$ a Taylor expansion is performed and new starting values are obtained for $G = 0^+$. It should be noted that this singularity at $G = 0$ is of the form $G^2 = a + b\xi$ with f finite. The integration is then carried out for $\Delta\xi < 0$ to $\xi = 0$. The only question that remains is whether, in fact, $G \to 0$ as ξ increases. Inspection of (7) for $G < 0$ reveals that for all $\alpha < 0$, $G \to 0$ as ξ increases. Hence, use of this integration scheme allows the following limits on α to be inferred:

$$\left.\begin{aligned} -\infty < \alpha < -1 &: \ f_w > 0, \text{ suction;} \\ \alpha = -1 &: \ f_w = 0; \\ -1 < \alpha < -\tfrac{1}{2} &: \ f_w < 0, \text{ blowing;} \\ \alpha = -\tfrac{1}{2} &: \ \text{blow-off;} \\ -\tfrac{1}{2} < \alpha < 0 &: \ G_w < 0; \\ \alpha \geqslant 0 &: \ \text{solution does not exist.} \end{aligned}\right\} \tag{14}$$

Thus, by use of the asymptotic solution, iteration is eliminated and every computer run can be considered a valid solution. Note that for any α within the described limits, once a solution is given, any f_w may be obtained by use of the group property.

Once a solution is computed a final integration is necessary to obtain the profiles as functions of η. In general, η may be written as

$$\eta = \int_0^\xi \frac{d\xi}{G}.$$

However, to integrate to the outer edge this must be altered as follows:

$$\eta = \int_0^{\xi_c - \epsilon_1} \frac{d\xi}{G} + \int_{\xi_c - \epsilon_1}^{\xi_c - \epsilon_2} \frac{d\xi}{G} + \int_{\xi_c - \epsilon_2}^{\xi_1} \frac{d\xi}{G}, \tag{15}$$

where ξ_c is the point where $G = 0$, ξ_1 is the initial value of ξ (at the outer edge), and ϵ_1/ξ_c, $\epsilon_2/\xi_c \ll 1$. The second integral in (15) may be evaluated about the singularity and yields

$$\int_{\xi_c - \epsilon_1}^{\xi_c - \epsilon_2} \frac{d\xi}{G} = -\frac{G^0}{\alpha\xi^2}, \tag{16}$$

where G^0 denotes the value of G at $\xi = \xi_c - \epsilon_1$. The numerical integration of (7) and (8) was performed using the Runge–Kutta integration scheme outlined by Gill (1951). In addition, a step-size criterion was programmed to allow the computer to choose its own step-size automatically. The solutions were carried out on a Bendix G–15 D computer where the running time per case was approximately 35 minutes.

Presentation and discussion of results

All the solutions were obtained with $f_\infty = 1$ and an initial value of $\xi = 0{\cdot}01$. Note again that the value of f_∞ is arbitrary and can be altered by use of the scale factor A. The suction results for various values of $\alpha < -1$ are presented in

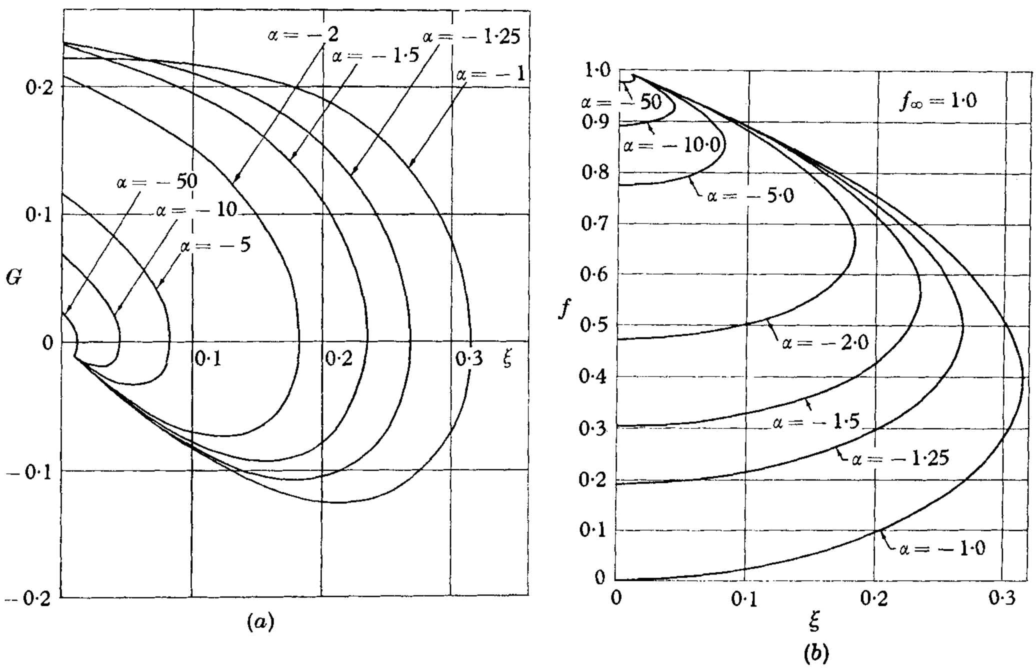
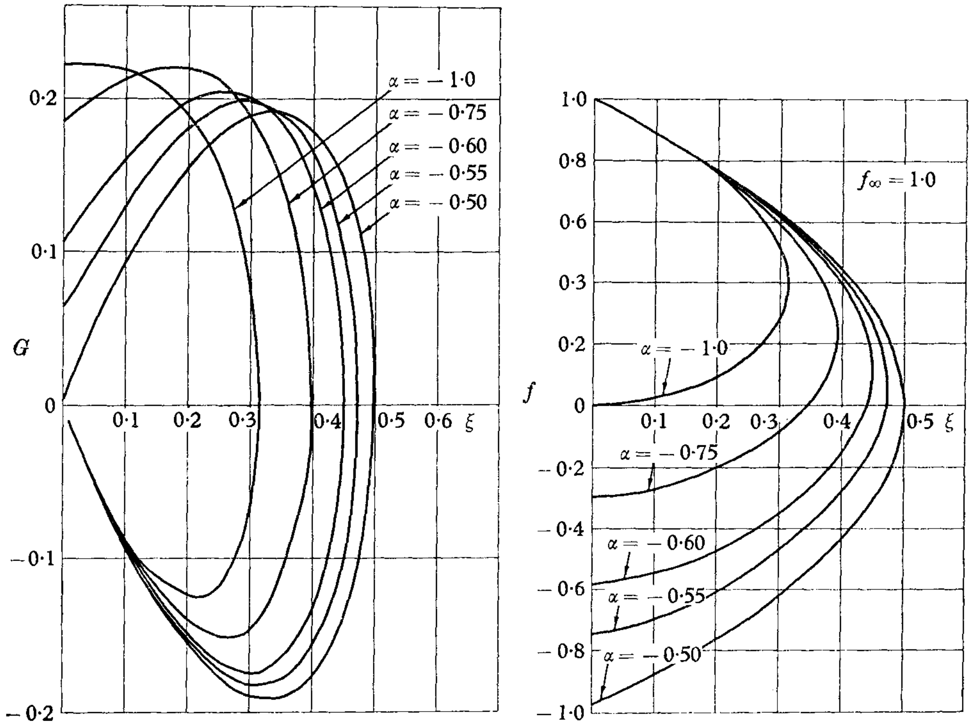


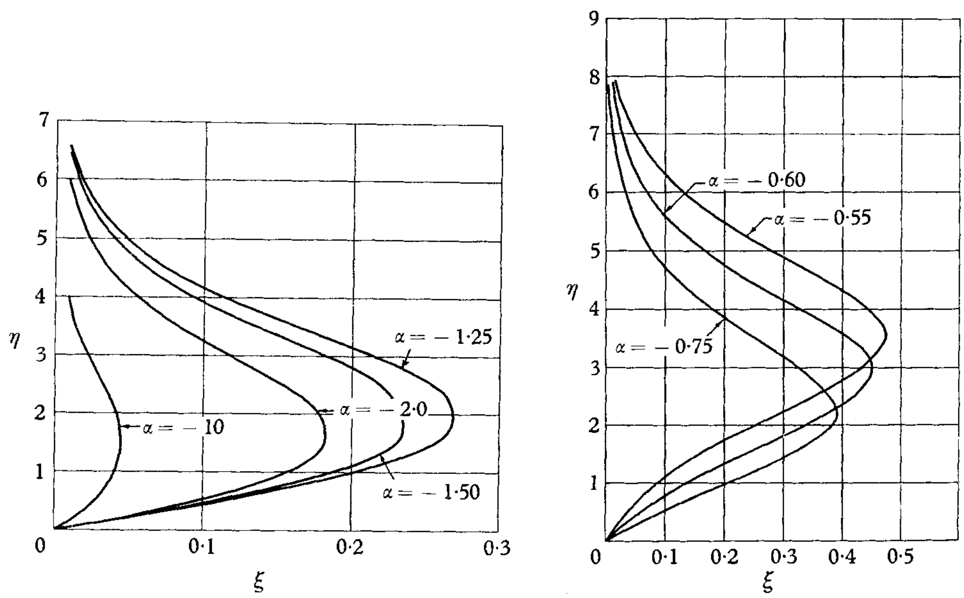
FIGURE 2. (*a*) Shear function with suction. (*b*) Stream function with suction.

figures 2 *a* and *b*; these give the dependent variables G and f *vs* the velocity ratio ξ. The results corresponding to injection, $-1 < \alpha < -\frac{1}{2}$, are presented in figures 3 *a* and *b*. It should be noted that while the computer can obtain a solution with $\alpha = -\frac{1}{2}$ corresponding to blow-off, the transformation to η becomes indeterminate, since at the wall $G = 0$. It is interesting, however, to see the profiles obtained from the computer run; f and G as obtained are also shown in figures 3 *a* and *b*. Some typical velocity profiles as functions of η, computed by (15), are shown in figures 4 *a* and *b*. With $f_\infty = 1{\cdot}0$, the eigenvalues of the resulting solutions f_w and G_w are shown in figure 5 and listed in table 1. In contradistinction to the results for a flat plate, it is noted that as f_w increases positively corresponding to $(-\alpha) \gg 1$, the shear parameter G_w reaches a maximum and then decreases. This can be thought of as due to the decrease of the maximum velocity which dominates the decrease in boundary-layer thickness due to the imposed suction. These two effects are clearly seen from the velocity profiles presented in figures 4*a* and *b*.



(*a*) (*b*)

FIGURE 3. (*a*) Shear function with injection. (*b*) Stream function with injection.



(*a*) (*b*)

FIGURE 4. (*a*) Velocity profiles with suction. (*b*) Velocity profiles with injection.

	$-\alpha$	f_w	G_w
Suction	50·0	0·976	0·0235
	10·0	0·892	0·0688
	5·0	0·776	0·117
	2·0	0·472	0·208
	1·5	0·305	0·232
	1·25	0·193	0·235
Impermeable wall	1·0	0·0	0·222
Injection	0·75	−0·298	0·186
	0·70	−0·379	0·159
	0·60	−0·591	0·106
	0·55	−0·749	0·0607
	0·50	−1·0	0·0

TABLE 1. Injection rate and shear parameter for $f_\infty = 1{\cdot}0$.

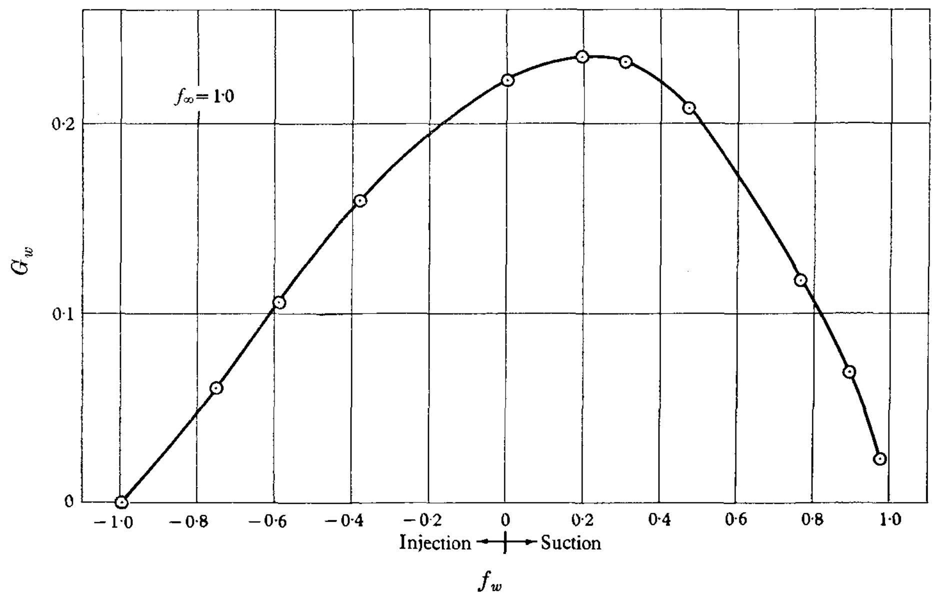


FIGURE 5. Variation of shear function with suction and injection.

It is pertinent now to obtain relationships for some of the important physical parameters. If a uniform reference flow is assumed then, from (2*b*),†

$$(\tilde{s}/\tilde{s}_c) = (r/r_c)^{(2k+1)/(1-\alpha)} \tag{17}$$

and

$$\tilde{s}_c = [(1-\alpha)(\rho_r \mu_r u_{rc}/(2k+1)]\, r_c^{2k+1}, \tag{18}$$

where u_r is evaluated from the definition of α and is found to be

$$(u_r/u_{r_c}) = (\tilde{s}/\tilde{s}_c)^{\alpha}. \tag{19}$$

† It should be pointed out that the r is to be interpreted as the distance along the wall for two-dimensional flows, $k = 0$, or as the radial distance from the jet for axisymmetric flows, $k = 1$.

Here the subscript c refers to values at a reference station s_c. By (17), (19) can be rewritten in terms of r as

$$(u_r/u_{r_c}) = (r/r_c)^{\alpha(2k+1)/(1-\alpha)} \tag{20}$$

and the velocity in jet can be shown to be

$$(u/u_{r_c}) = f'(r/r_c)^{\alpha(2k+1)/(1-\alpha)}. \tag{21}$$

From the transformation given by $(2a, b)$ it can be shown that the y-velocity component at the wall, w_w, is given by

$$w_w = -\mu_w u_r r^k (2\bar{s})^{-\frac{1}{2}} f(0).$$

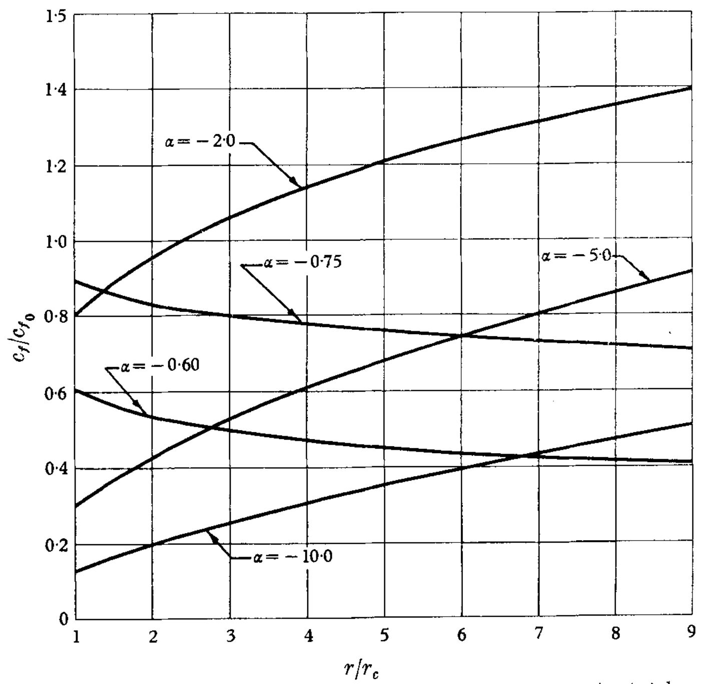


FIGURE 6. Distribution of skin friction. $C_{f_0} = P_2 \dfrac{\sqrt{2}}{9}\left(\dfrac{r}{r_c}\right)^{\frac{1}{4}}$.

Using (17), (18) and (20), this reduces to

$$w_w = -P_1 f(0)\,(1-\alpha)^{-\frac{1}{2}}\,(r/r_c)^{\{\frac{1}{2}(2k+1)(2\alpha-1)/(1-\alpha)\}+k}, \tag{22}$$

where $$P_1 = \mu_w u_{r_c}[(2k+1)/2\rho_r \mu_r u_{r_c} r_c]^{\frac{1}{2}}.$$

The skin friction coefficient, c_f, is given by

$$c_f = P_2 G(0)\,(1-\alpha)^{-\frac{1}{2}}\,(r/r_c)^{-(2\alpha+2k-1)/2(1-\alpha)}, \tag{23}$$

where $P_2 = [2\mu_r(2k+1)/\rho_r u_{r_c}]^{\frac{1}{2}} r_c^{(1-2k)/2}$. With $k = 1$ (axisymmetric flow) and various values of α, c_f/c_{f_0} was computed from (23), where c_{f_0} is the skin friction obtained for zero mass transfer, $\alpha = -1, f_w = 0$. This is shown in figure 6. As α decreases below $-\frac{1}{2}$ the skin friction first increases and then decreases significantly for the larger rates of suction and for reasonable distances downstream.

It is pointed out, however, that for suction c_f/c_{f_0} increases monotonically with r/r_c and that, as α decreases, the exponent increases to a limit of $+1$ for $k = 1$. Hence, at some large finite value of r/r_c the skin friction is larger for the larger rates of suction.

If incompressible flow is assumed, the boundary-layer thickness δ can be shown to be

$$\delta = \eta_e(1-\alpha)^{\frac{1}{2}} P_3(r/r_c)^{\{\frac{1}{2}(2k+1)(1-2\alpha)\}-k}, \tag{24}$$

where $P_3 \equiv [2r_c\mu_r/(2k+1)\rho_r u_{r_c}]^{\frac{1}{2}}$, and η_e is the value of η at the outer edge of the boundary layer and is chosen, say, when $\xi = 0{\cdot}001$ (at the outer edge).

3. The energy equation

Some features of similar thermal solutions, for Prandtl and Lewis number equal to unity, are now examined. The simplest solution is obtained for a very special case, namely, that in which $H_w = H_e = H_{ew}$, where H is the total enthalpy and the subscript e refers to conditions at the outer edge of the boundary layer. In this case the Crocco integral $H = A+Bu$ (with A and B constants) can be used to satisfy the boundary conditions, yielding

$$\frac{H-H_{ew}}{H_{m_c}-H_{ew}} = \left(\frac{r_c}{r}\right)^{-\alpha(2k+1)/(1-\alpha)} \frac{f'(n)}{f'_{\max}}, \tag{25}$$

where H_{m_c} is the value of H evaluated at an initial station, r_c, and at the point where $f' = f'_{\max}$. In this case the surface heat-transfer rate is given by

$$q_w/\tau_w = -(H_{m_c}-H_{ew})/u_{r_c}f'_{\max} \tag{26}$$

and by

$$(q_w/q_{w_c}) = (r/r_c)^{(-6\alpha k-4\alpha+1)/2(1-\alpha)}, \tag{27a}$$

where

$$-\frac{q_{w_1}r_cC_{p_w}}{(H_{m_c}-H_{ew})\lambda_w} = \left[\frac{2k+1}{2(1-\alpha)}\right]^{\frac{1}{2}} \frac{f''(0)}{f'_{\max}} \left(\frac{\rho_w u_{r_c} r_1}{\mu_w}\right)^{\frac{1}{2}}. \tag{27b}$$

Here τ_w represents wall shear stress and λ is the thermal conductivity.

For cases in which $H_e \neq H_w$ similar solutions can be derived by applying (2a) and (2b) to the usual boundary-layer energy equation and assuming that

$$H-H_e = (H_w-H_e)\,g(n), \quad H_w = H_w(\bar{s}), \quad H_e = \text{const.} \tag{28}$$

Then it follows that

$$g''+fg'-2Nf'g = 0, \quad \text{where} \quad N \equiv \frac{\bar{s}}{H_w-H_e}\frac{dH_w}{ds}. \tag{29}$$

Similar solutions are obtained from (29) only if

$$N = \text{const.} \tag{30}$$

A different solution for g (denoted by g_N) is obtained from (29) for each prescribed value of N. It can be noted that (29) is linear and any number of these solutions may be summed to form an additional solution. For each N, the following boundary conditions must be satisfied:

$$g(0) = 1{\cdot}0, \quad g(\infty) = 0. \tag{31}$$

Combination of (28), (29) and (30) leads to the following permissible form of non-isothermal surface enthalpy:

$$H_w = H_e + (H_w - H_e)_c\,(\tilde{s}/\tilde{s}_c)^N. \tag{32}$$

There is no restriction on the value of N; it may be positive, negative, zero and not necessarily an integer.

The case of constant surfrce temperature is given by $N = 0$, for which (29) becomes

$$g_0'' + fg_0' = 0, \tag{33}$$

which has the solution

$$g_0 - 1 = g_0'(0)\int_0^\eta \exp\left[-\int_0^\eta f d\eta\right] d\eta, \tag{34}$$

where
$$\frac{1}{g_0'(0)} = -\int_0^\infty \exp\left[-\int_0^\eta f d\eta\right] d\eta.$$

It can be shown that g_0 varies monotonically from 1 to zero. Therefore H varies monotonically from H_w to H_e. This severely restricts the régime of usefulness of this solution; the enthalpy profile cannot be prescribed arbitrarily at an initial station, but must be accepted as it is obtained under the postulate of similarity. Thus, one must exclude those cases in which the initial stagnation enthalpy of the wall jet is less than H_e or greater than H_w; at least up to the station where the enthalpy profile has deteriorated to an acceptable form. Furthermore, in this case heat must be passing from the surface into the stream when $H > H_e$. Note that these restrictions did not apply in the important special case ($H_e = H_w$) given previously. The case of the non-similar thermal field downstream of an arbitrary enthalpy profile can be treated by approximate, iterative or numerical techniques. However, the present purpose is to explore only the properties of the similar solutions.

In the general non-isothermal case the heat flux may be expressed in terms of similar solutions as follows:

$$-q_{w_N} = (\rho_w \mu_w u_r H_r r^k/(2\tilde{s})^{\frac{1}{2}})\, g_N'(0), \tag{35}$$

where
$$(q_w/q_{w_c})_N = (r_c/r)^\beta, \tag{36}$$

where
$$\beta = 2N\frac{2k+1}{2(1-\alpha)} - \frac{-2\alpha k - 2\alpha + 1}{2(1-\alpha)},$$

The quantity q_{w_c} can be obtained by evaluating (35) at the station $r = r_c$, q being the heat flux. For the non-isothermal case lengthy numerical procedures are required for the solution of the two-point boundary-value problem. However, for one particular value of N, namely, $N = -\frac{1}{2}$, a closed form solution is possible. For this case (29) reduces to

$$g'' + fg' + f'g = 0, \tag{37}$$

which can be integrated to give

$$g' + fg = 0, \tag{38}$$

where
$$g'(0) = -f_w. \tag{39}$$

Now (38) can be integrated to yield

$$g = \exp\left[-\int_0^\eta f\,d\eta\right], \quad g'(0) = -f_w. \tag{40}$$

For $N = -\frac{1}{2}$, the distribution of enthalpy along the surface is given by

$$H_w = H_e + (H_w - H_e)_1 (\tilde{s}_c/\tilde{s})^{\frac{1}{2}}. \tag{41}$$

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